Girsanov's Theorem

Preamble

You often meet a "Brownian Motion with drift" in Mathematical Finance. For example,

$$S_t = S_0 e^{\sigma W_t + (\mu - \sigma^2/2)t}$$

is a simple stock price model. The exponential term $\sigma W_t + (\mu - \sigma^2/2)t$ can be written

$$\sigma \left(W_t + \left(\frac{\mu}{\sigma} - \frac{\sigma}{2} \right) t \right)$$

and $W_t + \left(\frac{\mu}{\sigma} - \frac{\sigma}{2} \right) t$ is a 'BM with drift'.

The drift term, $\left(\frac{\mu}{\sigma} - \frac{\sigma}{2} \right) t$ in the example above, can be of many different forms. One that occurs often is

$$\int_0^t g(s)\, ds$$

where g is a process and $\mathbb{E}\left(\int_0^t g_s^2 ds \right) < \infty$.

So $\left(W_t - \int_0^t g_s ds \right)$ is a 'typical' BM with drift.

Of course, all of this assumes a measure, $\mathbb{P}$, in the background, under which (W_t) is a Brownian Motion. Girsanov's Theorem tells us that we can find a measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that $(W_t - \int_0^t g(s) ds)$ is a Brownian Motion under $\mathbb{Q}$. So $\mathbb{Q}$ looks at $(W_t - \int_0^t g(s) ds)$ and 'see's' a Brownian Motion. It is as if $\mathbb{Q}$ ignores the drift term, $\left(\int_0^t g(s)ds\right)$, and just looks at (W_t). But this is erroneous! $\mathbb{Q}$ looks at (W_t) and see's a Brownian Motion <u>with</u> drift. To prove Girsanov's result we need first of all to identify the measure $\mathbb{Q}$, and second to prove that $(W_t - \int_0^t g(s)ds)$ is a BM under $\mathbb{Q}$. For the second part we employ Levy's Theorem and we will have to use Theorem 0.?

<u>Theorem (Girsanov)</u>

Let $(\Omega, \mathcal{F}_\infty, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_t)$, $t \in [0,\infty)$, satisfying the usual conditions. Let (W_t) be a standard Brownian Motion. Let (g_t) be a measurable adapted process such that there is $K > 0$

$$|g(s,\omega)| \leq K \qquad 0 \leq s < \infty,\ \omega \in \Omega.$$

Let $\hat{W}_t = W_t - \int_0^t g_s ds$ and define for

$$T < \infty, \qquad \mathbb{Q}(E) = \int_E e^{\int_0^T g_s dW_s - \frac{1}{2}\int_0^T g_s^2 ds} \, d\mathbb{P}$$

for $E \in \mathcal{F}_T$. Then $(\hat{W}_t)$ is a $\mathbb{Q}$-Brownian Motion on $[0,T]$.

<u>Proof</u>

Since g is a bounded process, $\left(\int_0^t g_s dW_s\right)$ is a martingale while $\left(\int_0^t g_s ds\right)$ is a bounded variation process. So, writing $\int_0^t g_s dW_s - \frac{1}{2}\int_0^t g_s^2 ds = Z_t$

then (Z_t) is a semi-martingale. Let $f(x) = e^x$, then, using Itô's Lemma for semi-martingales,

!!!! I've written Z_s for e^{Z_s} in an earlier version!

$$e^{Z_t} = e^{Z_0} + \int_0^t e^{Z_s} g_s\, dW_s - \tfrac{1}{2}\int_0^t e^{Z_s} g_s^2\, ds + \tfrac{1}{2}\int_0^t e^{Z_s} d\langle Z\rangle_s$$

Now, $\langle Z\rangle_t = \langle Z, Z\rangle_t = \langle \int_0^t g\, dW, \int_0^t g\, dW\rangle = \int_0^t g_s^2\, ds.$

So as $Z_0 = 0$, and "$d\langle Z\rangle_s = g_s^2\, ds$", then

$$e^{Z_t} = 1 + \int_0^t e^{Z_s} g_s\, dW_s \qquad 0 \leq t \leq T.$$

Because g is bounded $\left(\int_0^t e^{Z_s} g_s\, dW_s\right)$ is a martingale and so therefore is (e^{Z_t}). Clearly $e^{Z_T} > 0$ $\mathbb{P}$.a.s. and therefore $\mathbb{Q}$ is a probability measure equivalent to $\mathbb{P}$. Note also,

$$\mathbb{Q}(E) = \int_E e^{Z_T}\, d\mathbb{P} = \int_E e^{Z_t}\, d\mathbb{P} \quad \text{for } E \in \mathcal{F}_t$$

this follows because (e^{Z_t}) is a martingale;

$$\int_E e^{Z_T}\, d\mathbb{P} = \int_\Omega e^{Z_T} I_E\, d\mathbb{P} = \int_\Omega M_t^{\mathbb{P}}(e^{Z_T} I_E)\, d\mathbb{P}$$

$$= \int_\Omega M_t^{\mathbb{P}}(e^{Z_T})\, I_E\, d\mathbb{P} = \int_E e^{Z_t}\, d\mathbb{P}.$$

This means our results for differing T are consistent. So, we have identified our measure $\mathbb{Q}$. Next we want to show that $(W_t - \int_0^t g_s\, ds)$ is a $\mathbb{Q}$ Brownian Motion. According to Levy's Theorem we should show;

(i) $(W_t - \int_0^t g_s\, ds)$ is a continuous process

(ii) $(W_t - \int_0^t g_s\, ds)$ is a $\mathbb{Q}$-martingale

(iii) $\left(W_t - \int_0^t g_s ds\right)$ has $\mathbb{Q}$ - quadratic variation t.

This last point is very important; the cross-variation $\langle X, Y\rangle$ of $\mathbb{P}$ - martingales, X and Y, depends on the measure $\mathbb{P}$! Changing the measure to $\mathbb{Q}$, will not stop $\langle X,Y\rangle$ from being a bounded variation process, but it may not see $XY - \langle X,Y\rangle$ as an L^1-martingale!

Item (i) is satisfied as (W_t) is a continuous process and so is the integral of g.

Item (ii). Recall theorem 0.5, we need to show that

$$\left(W_t - \int_0^t g_s ds\right) e^{Z_t} = \left(W_t - \int_0^t g_s ds\right)\left(1 + \int_0^t e^{Z_s} g_s dW_s\right)$$

is a $\mathbb{P}$ martingale. Using the product rule

$$\left(W_t - \int_0^t g_s ds\right) Z_t = \int_0^t \left(W_s - \int_0^s g_r dr\right) d e^{Z_s} + \int_0^t e^{Z_s} d\left(W_s - \int_0^s g_r dr\right) + \left\langle W - \int g_r dr, e^{Z}\right\rangle_t$$

Now, "$d e^{Z_s} = e^{Z} g dW$" and "$d\left(W_r - \int g_r dr\right) = dW_r - g_r dr$".

$$\text{So} \int_0^t \left(W_s - \int_0^s g_r dr\right) d e^{Z_s} = \underbrace{\int_0^t W_s e^{Z_s} g_s dW_s}_{\mathbb{P}\text{- martingale}} - \underbrace{\int_0^t \left(\int_0^s g_r dr\right) e^{Z_s} g_s dW_s}_{\mathbb{P}\text{- martingale}}$$

$$\int_0^t e^{Z_s} d\left(W_s - \int_0^s g_r dr\right) = \underbrace{\int_0^t e^{Z_s} dW_s}_{\mathbb{P}\text{- martingale}} - \int_0^t e^{Z_s} g_s ds$$

$$\left\langle W_r - \int g_r dr, 1 + \int e^{Z} g dW\right\rangle = \langle W, 1\rangle + \left\langle W, \int e^{Z} g dW\right\rangle - \left\langle \int g_r dr, 1\right\rangle - \left\langle \int g_r dr, \int e^{Z} g dW\right\rangle$$

any cross-variation with a bounded variation process is zero, leaving

$$\langle W-\int g\,dr,\ 1+\int e^{Z} g\,dW\rangle = \langle W, \int e^{Z} g\,dW\rangle$$
$$= \langle \int 1\,dW, \int e^{Z} g\,dW\rangle$$
$$= \int e^{Z_s} g_s\, d\langle W,W\rangle_s$$
$$= \int e^{Z_s} g_s\,ds$$

So this term cancels with one in $\int_0^t e^{Z_s} d\left(W_s - \int_0^s g\,dr\right)$.

All that is left are terms which are $\mathbb{P}$-martingales. So $(W_t - \int_0^t g_s\,ds)$ is a $\mathbb{Q}$ martingale.

Finally we must show that the $\mathbb{Q}$ variation of $(W_t - \int_0^t g_s\,ds)$ is " t ". Recall the fundamental result; " $(\langle X\rangle_t)$ is the (natural) bounded variation process such that $(X_t^2 - \langle X\rangle_t)$ is a martingale ". So we must show that

$((W_t - \int_0^t g_s\,ds)^2 - t)$ is a $\mathbb{Q}$ martingale, equivalently, the process,

$$\left((W_t - \int_0^t g_s\,ds)^2 - t\right)\left(1+\int_0^t e^{Z_s} g_s\,dW_s\right) \qquad t \geq 0$$

is a $\mathbb{P}$ martingale. We are going to need the product rule again!

First of all write $U_t = W_t - \int_0^t g_s\,ds$, then under $\mathbb{P}$,

$$U_t^2 = 2\int_0^t U_s\,dU_s + \langle U,U\rangle$$
$$= 2\int_0^t U_s\,dW_s - 2\int_0^t U_s g_s\,ds + \langle W,W\rangle_t$$

So $U_t^2 - t = 2\int_0^t U_s\,dW_s - 2\int_0^t U_s g_s\,ds$ since $\langle W,W\rangle_t = t$.

Using the product rule again, under $\mathbb{P}$,

$$(U_t^2 - t)\left(1+\int_0^t e^{Z_s} g_s \, dW_s\right) = \int_0^t (U_s^2 - s)\, e^{Z_s} g_s \, dW_s + 2\int_0^t e^{Z_s} U_s \, dW_s - 2\int_0^t U_s e^{Z_s} g_s \, ds$$
$$+ \left\langle 2\int_0^{t} U \, dW - 2\int_0^{t} U g \, ds \, , \, 1 + \int_0^{t} g e^{Z} dW \right\rangle$$

The term

$$\left\langle 2\int_0^t U \, dW - 2\int_0^t U g \, ds \, , \, 1+\int_0^t g e^{Z} dW \right\rangle = \left\langle 2\int_0^t U \, dW \, , \, \int_0^t g e^{Z} dW \right\rangle$$
$$= 2\int_0^t U_s e^{Z_s} g_s \, ds$$

and cancels! We are left with

$$\int_0^t (U_s^2 - s)\, e^{Z_s} g_s \, dW_s + 2\int_0^t U_s e^{Z_s} \, dW_s$$

which is a $\mathbb{P}$ martingale. So $\left(W_t - \int_0^t g_s \, ds\right)$ is a $\mathbb{Q}$, Brownian Motion.